Asymptotic Analysis of Thick Axisymmetric Turbulent Boundary Layers

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Introduction

THE similarity laws governing the axisymmetric turbulent boundary layer have been the subject of prolonged debate.[1] Herein constant pressure flow at velocity U past a circular cylinder of radius a with the axis aligned to the flow is considered. An asymptotic analysis of this flow has been presented earlier[1] in the limiting situation $a_+ = aU_\tau/\nu \to \infty$, $\delta/a = \mathcal{O}(1)$, where U_τ is the friction velocity, δ the boundary-layer thickness, and ν the kinematic viscosity. Some interesting experiments reported by Willmarth et al.[2] have δ/a as high as 42 and a_+ as low as 33.4. In the present Note the authors extend their earlier analysis to cover this wider range of parameters.

The asymptotic analysis[1] shows that the inner and outer velocity distributions would, in general, have the following functional dependences:

$$u_+ = u/U = u_+(X, y_+, 1/a_+, \epsilon/\delta_+, \epsilon_X/\delta_+) \tag{1}$$

$$u_- = (U-u)/U_\tau = u_-(X, y_-, \delta/a, 1/\delta_+, \epsilon_X/\epsilon) \tag{2}$$

where

$$X = \int^x \left(\frac{\epsilon^2}{\theta_l}\right) dx, \quad y_+ = \frac{yU_\tau}{\nu}, \quad y_- = \frac{y}{\delta}$$

$$\delta_+ = \frac{\delta U_\tau}{\nu}, \quad \epsilon = \frac{U_\tau}{U}, \quad \epsilon_X = \frac{\partial \epsilon}{\partial X}$$

θ_l is a characteristic momentum length, and x and y streamwise and normal coordinates, respectively. In the limit of large a_+ and δ/a of order unity, it was argued[1] that Eqs. (1) and (2) would simplify to

$$u_+ = u_+(y_+, 1/a_+) + \mathcal{O}(1/a_+), \quad u_- = u_-(y_-, \delta/a) + \mathcal{O}(\epsilon) \tag{3}$$

leading to the inertial sublayer (as $y_+ \to \infty$, $y_- \to 0$)

$$u_+ = A\log y_+ + B(a_+^{-1}), \quad u_- = -A\log y_- + D(\delta/a) \tag{4}$$

in which slope A is independent of both a_+ and δ/a and, therefore, necessarily the same as in two-dimensional flow; but intercept B in the inner law depends on a_+ and, similarly, intercept D in the outer law on δ/a. A detailed review of the available data, the bulk of which is at $\delta/a \le 2$, showed that the conclusions of the asymptotic theory were largely supported by experiments.

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Analysis

The earlier analysis needs to be modified if a_+ is not large but δ/a is considerably in excess of unity. If $\delta/a \gg \epsilon$, then the dominant parameter in the inner layer [Eq. (1)] is $1/a_+$. Furthermore, if in an appropriate limit we require

$$\delta/a = \delta_+/a_+ = \mathcal{O}(a_+) \quad \text{or} \quad \delta_+ = \mathcal{O}(a_+^2) \tag{5}$$

(this limit is relevant for some of the data sets of Ref. 2), then we may expect that in an extension to higher orders of the outer law [Eq. (3)] we need to carry δ_+ (or a_+) as a parameter in addition to δ/a. Using the arguments of Ref. 1 we have

$$\epsilon^{-1} \sim \ln\delta_+; \quad \frac{\epsilon_X/\epsilon}{\delta/a} = \frac{\epsilon}{\delta_+}\,\frac{a}{\delta}\,\frac{d\delta_+}{dX} \tag{6}$$

It follows that if $\delta/a \gg \epsilon$, then $\epsilon_X/\epsilon \ll \delta/a$. Therefore, the relevant extensions of Eq. (3) are

$$u_+ = u_+(y_+; 1/a_+), \quad u_- = u_-(y; \delta/a, 1/a_+) \tag{7}$$

Millikan's argument[1] now leads to the inertial sublayer

$$u_+ = A\log y_+ + B, \quad u_- = -A\log y_- + D \tag{8}$$

where the functional dependences are

$$A = A(1/a_+), \quad B = B(1/a_+), \quad D = D(\delta/a, 1/a_+) \tag{9}$$

Thus, this extended asymptotic theory permits a dependence of both the Karman parameter A and the outer intercept D on a_+. The implied dependence of the outer flow on the viscosity (through a_+) may raise questions about whether the flow still can be characterized as fully turbulent.

Comparison with Experiment

Due to space limitations, only some of the measured velocity profiles[2] are shown plotted in the usual inner and outer variables in Figs. 1 and 2. It is seen that a log law exists in all data sets, but the slope (or Karman parameter) does not remain constant. Figure 3a shows this variation as a function of a_+. From the data available, it is not possible to determine the

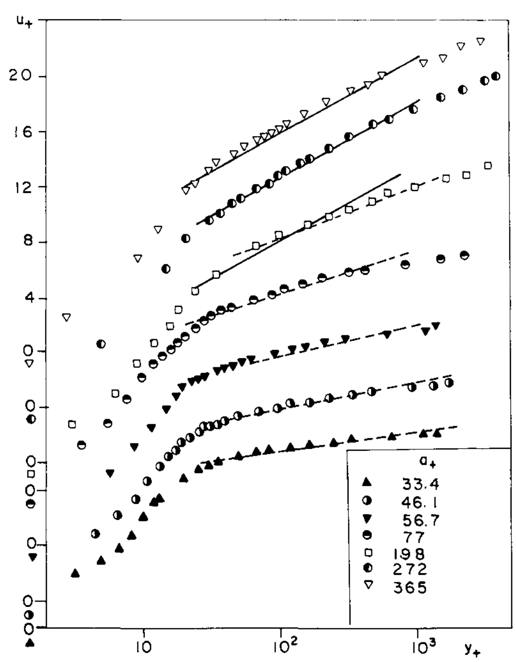

Fig. 7 Measured velocity profiles[2] in wall variables.

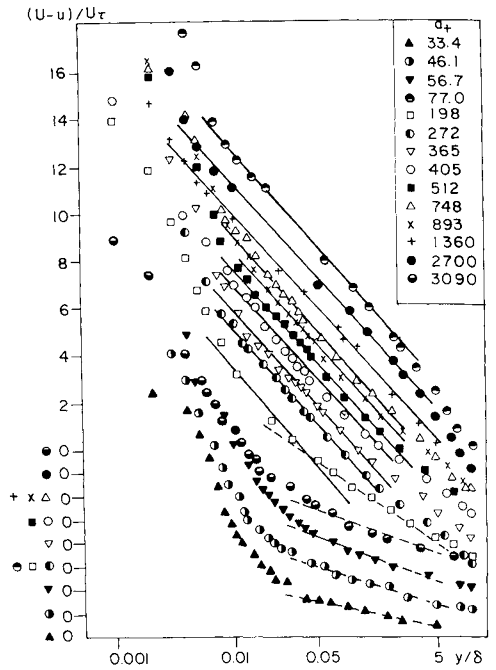

Fig. 2 Measured velocity profiles[2] in outer ("defect") variables.

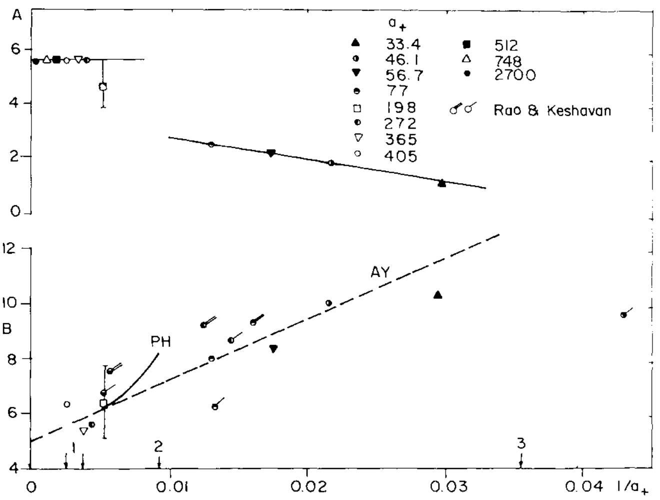

Fig. 3 Variation of log law parameters: a) slope or Karman parameter, b) intercept. Arrows on the abscissa mark the following values (from Ref. 6) for pipe flow. 1—$a_+ = 315$-435, above which log law with universal constants is valid. 2—$a_+ = 110$, corresponding to the upper limit of transition and disappearance of intermittency. 3—$a+ \simeq 28$, critical value below which relaminarization is suggested. Experimental data from Ref. 2 unless otherwise stated, in which case they are from sources cited in Ref. 1.

precise value of a_+ at which A departs from its classical value ($\simeq 5.6$), but one can say that there is no such departure for $a_+ \gtrsim 250$. Thereafter, A decreases rapidly with a_+, reaching a low value of 1.2 at $a_+ \simeq 33$.

Figure 3b shows the inner intercept as a function of a_+; the various other data and the relation of Afzal and Yajnik for pipe flow,[1]

$$B = 5 + 236/a_+ \tag{10}$$

are also shown. If the value of B for a flat plate is taken to be 5.0,[3] the variation [Eq. (10)] may be considered valid down to $a_+^{-1} \rightarrow 0$; if, on the other hand,[4] $B \simeq 5.45$ is taken, it would be more reasonable to take B as constant for $a_+ \gtrsim 250$.

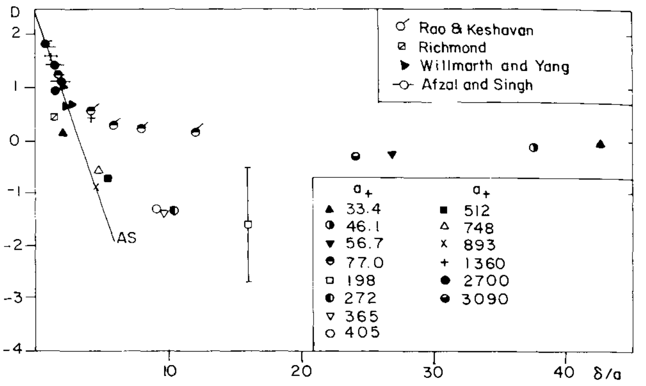

Fig. 4 Variation of intercept in velocity defect law. (For data sources see Fig. 3.)

Figure 4 shows the outer intercept D with δ/a as the abscissa. For $\delta/a \lesssim 3$ all available data appear to be in reasonable agreement with the variation suggested in Ref. 5:

$$D = 2.5 - 7.2\delta/a \tag{11}$$

The present asymptotic analysis indicates that as a_+ decreases we may expect a family of curves in Fig. 4 with a_+ as a parameter. However, in the only set of measurements available at large δ/a, there is no *independent* variation of a_+, so that at present one cannot sketch this possible dependence of D on a_+. Also, however, it is clear that at large a_+ the curves tend to collapse, leading to a variation of D with δ/a as in Eq. (11); and possibly that at much lower a_+ (~ 50) D nearly vanishes.

Patel[6] has suggested that below a critical value of $a_+ \simeq 28$ the axisymmetric turbulent boundary layer will revert to a laminar state; but there is no direct evidence of this from experiment.[2,5] Nevertheless, we note that at large δ/a the production of turbulent energy, always a sharp maximum near the wall, occurs predominantly over a cylinder whose area is proportional to a, and has to sustain turbulence over a volume proportional to δ^2. In other terms, an axisymmetric turbulent boundary layer experiences a surface shear force that is approximately (a/δ) times the force that sustains a two-dimensional turbulent boundary layer of the same thickness. Therefore, one may legitimately expect (at the very least) anomalous behavior at high δ/a or low a_+ as a consequence of inadequate energy "supply" from the surface. Also, as $a/\delta \rightarrow 0$, the boundary layer must begin to behave like an axisymmetric wake, which must laminarize sufficiently far downstream as its Reynolds number keeps falling. Such laminarization would be an instance of "dissipative" reversion,[7] which tends to be very slow, and, therefore, its effects may escape immediate detection.

Conclusions

An axisymmetric turbulent boundary layer is effectively plane if $a_+ \gtrsim 250$, $\delta/a \lesssim 1$. Even outside this range the log law holds in the overlap sublayer, but for smaller values of a_+ the Karman parameter A and inner intercept B both depend on a_+; A reaching values as low as 1.2 at $a_+ \simeq 30$, around which the flow probably ceases to be fully turbulent. The outer intercept depends only on δ/a for $\delta/a \lesssim 3$, could depend on a_+ also at higher values, and seems to become very small at large δ/a.

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Perturbations of a Transonic Flow with Vanishing Shock Waves

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Introduction

THE computation of transonic flows using potential theory is fairly commonplace, although the computation cost is still high and, consequently, considerable effort has been expended in trying to reduce the total costs of the necessary calculations. One approach is to improve the numerical algorithms so that the central processor time for a given computation is reduced. A second approach is to attempt to extend the usefulness of a relatively few expensive high-grade numerical computations by extracting as much information as possible from the computed results. This second approach is the driving force behind the development of the transonic perturbation theory.[1,2] A consequence of the theory is that nonlinear transonic solutions can be interpolated or extrapolated to give a range of solutions if only two solutions are initially known.

The main problem in developing a perturbation theory for transonic flow with a shock wave is that the basic premise of small perturbations breaks down if the shock wave changes location. In Ref. 1 a strained coordinate technique is developed in which the shock location is kept invariant in the new coordinate system. A restriction of the theory is that shock waves cannot be generated or destroyed during the perturbation. The perturbation equations are linear in the strained coordinates but on the return to physical coordinates the correct nonlinearity of the problem is apparent. One advantage of the linearity of the perturbation equation is that superposition of the effects of perturbations in several different parameters is possible, a feature which is very useful in design applications. Although the utility of the basic perturbation theory and its unsteady development, the indicial method, has been proven, the restriction that shock waves cannot be generated or destroyed during the perturbation invalidates the application of the theory to several important areas in both steady and unsteady transonic aerodynamics. This Note concerns the results of a preliminary investigation[3] into the behavior of the perturbation theory in areas where the shock wave will vanish during a small perturbation.

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Analysis

The basic equation used in the present investigation is the transonic small disturbance equation, which is written in the form[4]

$$\phi_{xx} + \phi_{yy} = \phi_x \phi_{xx} \tag{1}$$

where ϕ is a scaled perturbation velocity potential related to the perturbation velocity potential. In this version of the transonic small disturbance equations, sonic conditions exist when

$$u = \phi_x = 1 \tag{2}$$

The usual boundary conditions for Eq. (1) are applied. Equation (1) and its boundary conditions can be written in integral form[3] to give

$$u - (u^2/2)\cos^2\sigma_s = I_L + I_s = F \tag{3}$$

where I_L is a known function of the tangency boundary conditions and I_s a surface integral. It is not necessary to know the precise form of I_L and I_s except to note that both are continuous functions of (x,y) and have continuous first derivatives; thus, the detailed definition is not given here. The angle the shock makes with the y axis is σ_s.

Equation (3) must be solved subject to the following regularity conditions which eliminate expansion shocks.

$$\left(\frac{\partial F}{\partial x}\right)_{x_0} = 0 \tag{4a}$$

$$(F)_{x_0} = 1/2m^2 \tag{4b}$$

Here, $x = x_0$ is the switching point for a zero strength jump and $m = \cos\sigma_s$. If Eqs. (4a) and (4b) are also satisfied at a decelerating point in the flow, then there is no compression shock and the flow is shock-free. A detailed description of the integral equation method is given in Ref. 4.

The rationale and detailed description of the transonic perturbation method are given by Nixon.[1,2] The basic integral equations for the perturbation problem are as follows.

$$u_1(1-u_0) = I_{L_1} + I_{s_1}(u_1,u_0) + \delta x_s I_f(u_0) \tag{5}$$

where, if x' is a strained coordinate, given by

$$x = x' + \delta x_s x_1(x')$$

where $x_1(x')$ is a specified straining function, then

$$u_1(x',y) = \phi_{1x'}(x',y)$$

$$u_0(x',y) = \phi_{0x'}(x',y)$$

I_{L1} is a known continuous function of the perturbation boundary conditions and I_f is a known function of u_0. The quantities I_{s1} and I_f are not required in detail in the present work and the reader is referred to Ref. 1 for further information. The subscript *1* denotes a perturbation quantity.

Equation (5) must be solved subject to the following regularity condition.

$$[I_{L_1} + I_{s_1} + \delta x_s I_f(u_0)]_{x=x_0} = 0 \tag{6}$$

where $x = x_0$ denotes the location of the switching point. Equation (6) is sufficient to give δx_s. The main restriction on the perturbation theory is that shock waves cannot be generated or destroyed during a perturbation.